THE KÄHLER-RICCI FLOW AND THE $\bar\partial$ OPERATOR ON VECTOR FIELDS[1]

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Abstract

The limiting behavior of the normalized Kähler-Ricci flow for manifolds with positive first Chern class is examined under certain stability conditions. First, it is shown that if the Mabuchi K-energy is bounded from below, then the scalar curvature converges uniformly to a constant. Second, it is shown that if the Mabuchi K-energy is bounded from below and if the lowest positive eigenvalue of the $\bar\partial^\dagger\bar\partial$ operator on smooth vector fields is bounded away from 0 along the flow, then the metrics converge exponentially fast in C^∞ to a Kähler-Einstein metric.

1 Introduction

The Kähler-Ricci flow is arguably the most natural non-linear heat flow on a Kähler manifold, and its singularities and asymptotic behavior can be expected to provide a particularly deep probe of the geometry of the underlying manifold. For manifolds X with positive first Chern class, the Kähler-Ricci flow exists for all times [C], and the issue is its asymptotic behavior. The convergence of the flow would produce a Kähler-Einstein metric, the existence of which had been conjectured by Yau [Y2] to be equivalent to the stability of X in the sense of geometric invariant theory. Thus the convergence and, more generally, the asymptotic behavior of the flow should be related to stability conditions.

There have been however only relatively few results in this direction. In fact, the convergence of the flow for $c_1(X) > 0$ has been established only for $X = \mathbf{CP}^1$ [H, Ch, CLT], for X admitting a metric with positive bisectional curvature (and hence must be $\mathbf{CP}^n$) [CT], under the assumption that X already admits a Kähler-Einstein metric [P2] or a Kähler-Ricci soliton [TZ2], and for X toric with vanishing Futaki invariant [Z] (which is known to imply that X admits a Kähler-Einstein metric [WZ]). In [PS], it was shown that certain stability conditions do imply the convergence of the flow, without an a priori assumption on the existence of a Kähler-Einstein metric or a Kähler-Ricci soliton, but with an additional assumption on curvature bounds.

The purpose of this paper is to relate the asymptotic behavior of the Kähler-Ricci flow to stability conditions, without either of the previous assumptions of curvature bounds or

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existence of Kähler-Einstein metrics or Kähler-Ricci solitons. More specifically, we deal with two types of stability conditions. The first is the familiar lower boundedness of the Mabuchi K-energy [B, T, D2, PS]. The second is the lower boundedness of the first positive eigenvalue of the $\bar\partial^\dagger\bar\partial$ operator on smooth $T^{1,0}$ vector fields. This second condition appears to be new, but it should be closely related to the stability condition (B) introduced in [PS], namely that the closure of the orbit of the complex structure J of X does not contain any complex structure $\tilde{J}$ with a strictly higher number of independent holomorphic vector fields.

Our results are of two types. To state them precisely, let X be a compact Kähler manifold of dimension n with $c_1(X)>0$, and let the Kähler-Ricci flow[1] be defined by

$$\frac{\partial}{\partial t} g_{\bar{k}j} = -(R_{\bar{k}j} - g_{\bar{k}j}), \qquad g_{\bar{k}j}|_{t=0} = (g_0)_{\bar{k}j}, \tag{1.1}$$

where $(g_0)_{\bar{k}j}$ is a given initial metric, with Kähler form $\omega_0 = \frac{\sqrt{-1}}{2}(g_0)_{\bar{k}j} dz^j \wedge d\bar{z}^k \in \pi\, c_1(X)$. The Mabuchi K-energy $\mathcal{M}(\omega_\phi) = \mathcal{M}(\phi)$ is the functional defined on $\pi c_1(X)$ by its value at some fixed reference point and its variation

$$\delta\mathcal{M}(\phi) = -\frac{1}{V}\int_X \delta\phi(R-n)\omega_\phi^n, \qquad V \equiv \int_X \omega_\phi^n = \pi^n c_1(X)^n, \tag{1.2}$$

where $\omega_\phi = \omega_0 + \frac{\sqrt{-1}}{2}\partial\bar\partial\phi \in \pi c_1(X)$ has been identified with its potential ϕ (modulo constants) and $R = g^{j\bar{k}}R_{\bar{k}j}$ denotes the scalar curvature of ω_ϕ. The first type of result assumes only a lower bound of the Mabuchi K-energy. Under such an assumption, using the continuity method, Bando [B] had shown the existence of Kähler metrics in $\pi c_1(X)$ with $||R-n||_{C^0}$ arbitrarily small. In [PS], §6, it was shown that, under the same assumption, $||R-n||_{L^2} \to 0$ for the Kähler-Ricci flow. Here we show:

Theorem 1 *Assume that the Mabuchi K-energy is bounded from below on the Kähler class $\pi c_1(X)$. Let $g_{\bar{k}j}(t)$ be any solution of the Kähler-Ricci flow (1.1), and let $R(t)$ be the scalar curvature of $g_{\bar{k}j}(t)$. Then we have*

(i) $\|R(t)-n\|_{C^0} \to 0$ *as* $t\to\infty$;

(ii) $\displaystyle\int_0^\infty \|R(t)-n\|_{C^0}^p\, dt < \infty, \qquad$ *when* $p>2$.

In view of Lemma 6 below, the gap between a lower bound for the Mabuchi K-energy and the existence of a Kähler-Einstein metric is thus at most the gap between $L^p[0,\infty)$, $p>2$, and $L^1[0,\infty)$. The second type of result assumes both a lower bound of the Mabuchi K-energy and a stability condition (S):

[1]In this paper, we consider only the normalized Kähler-Ricci flow, and designate this flow simply by "Kähler-Ricci flow".

Theorem 2 *Fix $\omega_0 \in \pi c_1(X)$. Let $g_{\bar{k}j}(t)$ be the solution of the Kähler-Ricci flow with initial value $(g_0)_{\bar{k}j}$, and $\omega(t)$ the corresponding Kähler forms. Let λ_ω be the lowest strictly positive eigenvalue of the Laplacian $\bar{\partial}^\dagger\bar{\partial} = -g^{j\bar{k}}\nabla_j\nabla_{\bar{k}}$ acting on smooth $T^{1,0}$ vector fields.*

(i) If the following two conditions are satisfied,

$$\begin{array}{ll} \text{(A)} & \inf_{\omega\in\pi c_1(X)}\mathcal{M}(\omega) > -\infty \\ \text{(S)} & \inf_{t\in[0,\infty)}\lambda_{\omega(t)} > 0 \end{array}$$

then the metrics $g_{\bar{k}j}(t)$ converge exponentially fast in C^∞ to a Kähler-Einstein metric.

(ii) Conversely, if the metrics $g_{\bar{k}j}(t)$ converge in C^∞ to a Kähler-Einstein metric, then the conditions (A) *and* (S) *are satisfied.*

(iii) In particular, if the metrics $g_{\bar{k}j}(t)$ converge in C^∞ to a Kähler-Einstein metric, then they converge exponentially fast in C^∞ to this metric.

As an immediate consequence, if the Mabuchi K-energy is bounded below on $\pi c_1(X)$ and

$$\inf_{\omega\in\pi c_1(X)}\lambda_\omega > 0$$

then Theorem 2 implies that every solution $g_{\bar{k}j}(t)$ of the Kähler-Ricci flow converges exponentially fast in C^∞ to a Kähler-Einstein metric.

We conclude this introduction with some remarks on the proof. The recent works of Perelman [P2] have provided powerful tools for the study of the Kähler-Ricci flow, including a non-collapsing theorem and the uniform boundedness of the Ricci potential and of the scalar curvature. On the other hand, there are still no known uniform bounds for the Riemannian and the Ricci curvatures. We bypass this difficulty by exploiting two features of the flow: the first is its parabolicity, so that certain stronger norms for the key geometric quantities can be controlled by weaker norms at an *earlier* time (e.g. Lemma 1); and the second is that such bounds at earlier times can still produce the desired convergence statements when combined with suitable *differential-difference inequalities* (see e.g. the inequality (5.5) below).

2 Perelman's results

Perelman [P1], [P2] proved the following estimates for a solution of (1.1)(see [ST] for a detailed exposition). The first is bounds for the Ricci potential $u = u(t)$ defined by

$$R_{\bar{k}j} - g_{\bar{k}j} = -\partial_j\partial_{\bar{k}}u, \qquad \frac{1}{V}\int_X e^{-u}\omega^n = 1, \tag{2.1}$$

and the second is a non-collapsing theorem:

(i) There exists a constant C_0 depending only on $g_{\bar{k}j}(0)$ such that

$$\|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R\|_{C^0} \leq C_0. \tag{2.2}$$

(ii) Let $\rho > 0$ be given. Then there exists $c > 0$ depending only on $g_{\bar{k}j}(0)$ and ρ such that for all points $x \in X$, all times $t \geq 0$ and all r with $0 < r \leq \rho$, we have

$$\int_{B_r(x)} \omega^n > c\, r^{2n}, \tag{2.3}$$

where $B_r(x)$ is the geodesic ball of radius r centered at x with respect to the metric $g = g(t)$.

For the reader's convenience, we note that the exact statement of (ii) can be derived from (i) as follows. Make a change of variable $t = -\log(1-2s)$ and define a Riemannian metric $h = h(s)$ by $h(s) = (1-2s)g(t(s))$. Then $h(s)$ is a solution of Hamilton's Ricci flow for $s \in [0, 1/2)$, and using the scalar curvature bound of (i), one can apply Theorem 8.3.1 of [To], or the arguments contained in [ST].

3 A smoothing lemma

The important idea of a smoothing lemma, exploiting the parabolicity of the Kähler-Ricci flow, is due to Bando [B]. For our purposes, we need the version below, the key feature of which is the fact that it does not require a lower bound on the Ricci curvature:

Lemma 1 *There exist positive constants δ and K depending only on n with the following property. For any ε with $0 < \varepsilon \leq \delta$ and any $t_0 \geq 0$, if*

$$\|u(t_0)\|_{C^0} \leq \varepsilon,$$

then

$$\|\nabla u(t_0+2)\|_{C^0} + \|R(t_0+2) - n\|_{C^0} \leq K\varepsilon.$$

Proof. By making a translation in time we can assume, without loss of generality, that $t_0 = 0$. It is well-known that u evolves by

$$\frac{\partial}{\partial t} u = \Delta u + u - b, \tag{3.1}$$

where $b = b(t)$ is the average of u with respect to the measure $e^{-u}\omega^n$:

$$b = \frac{1}{V}\int_X u e^{-u}\omega^n. \tag{3.2}$$

It is convenient to define a new constant $c = c(t)$ for $t \geq 0$ by $\dot{c} = b + c$, $c(0) = 0$. Then set $\hat{u}(t) = -u(t) - c(t)$. We have $\|\hat{u}(0)\|_{C^0} \leq \varepsilon$ and $\hat{u}$ evolves by

$$\frac{\partial}{\partial t}\hat{u} = \Delta\hat{u} + \hat{u}. \tag{3.3}$$

Following [B], we calculate

$$\frac{\partial}{\partial t}\hat{u}^2 = \Delta\hat{u}^2 - 2|\nabla\hat{u}|^2 + 2\hat{u}^2 \tag{3.4}$$

$$\frac{\partial}{\partial t}|\nabla\hat{u}|^2 = \Delta|\nabla\hat{u}|^2 - |\nabla\overline{\nabla}\hat{u}|^2 - |\nabla\nabla\hat{u}|^2 + |\nabla\hat{u}|^2 \tag{3.5}$$

$$\frac{\partial}{\partial t}\Delta\hat{u} = \Delta(\Delta\hat{u}) + \Delta\hat{u} + |\nabla\overline{\nabla}\hat{u}|^2. \tag{3.6}$$

Then we see from (3.4) that $\|\hat{u}(t)\|_{C^0} \leq e^2\varepsilon$ for $t \in [0, 2]$. From (3.5) we have

$$\frac{\partial}{\partial t}\left(e^{-2t}(\hat{u}^2 + t|\nabla\hat{u}|^2)\right) \leq \Delta\left(e^{-2t}(\hat{u}^2 + t|\nabla\hat{u}|^2)\right), \tag{3.7}$$

giving $\|\nabla\hat{u}\|^2_{C^0}(t) \leq e^4\varepsilon^2$ for $t \in [1, 2]$.

We will now prove a lower bound for $\Delta\hat{u}$. Set

$$H = e^{-(t-1)}(|\nabla\hat{u}|^2 - \varepsilon n^{-1}(t-1)\Delta\hat{u})$$

and compute using (3.5) and (3.6),

$$\frac{\partial}{\partial t}H \quad = \quad \Delta H - e^{-(t-1)}\left(\varepsilon n^{-1}\Delta\hat{u} + (1 + \varepsilon n^{-1}(t-1))|\nabla\overline{\nabla}\hat{u}|^2 + |\nabla\nabla\hat{u}|^2\right). \tag{3.8}$$

For $t \in [1, 2]$, using the inequality $(\Delta\hat{u})^2 \leq n|\nabla\overline{\nabla}\hat{u}|^2$ we obtain

$$\frac{\partial}{\partial t}H \quad \leq \quad \Delta H + e^{-(t-1)}n^{-1}(-\Delta\hat{u})(\varepsilon + \Delta\hat{u}). \tag{3.9}$$

We claim that $H < 2e^4\varepsilon^2$ for $t \in [1, 2]$. Otherwise, at the point $(x', t') \in X \times (1, 2]$ when this inequality first fails we have $-\Delta\hat{u} \geq e^4\varepsilon$. But since $(\frac{\partial}{\partial t} - \Delta)H \geq 0$ at this point, we also have $\varepsilon + \Delta\hat{u} \geq 0$, which gives a contradiction. Hence at $t = 2$ we have $H < 2e^4\varepsilon^2$ and

$$\Delta\hat{u} > -2ne^5\varepsilon,$$

on X.

By considering the quantity

$$K = e^{-(t-1)}(|\nabla\hat{u}|^2 + \varepsilon n^{-1}(t-1)\Delta\hat{u}),$$

we can similarly prove that $\Delta\hat{u} < 2ne^5\varepsilon$ at $t = 2$ (see [B]). Since $\Delta\hat{u} = R - n$ this completes the proof of the lemma. Q.E.D.

Remarks

(i) In the statement of the lemma, (t_0+2) could be replaced by $(t_0+\zeta)$ for any positive constant ζ, at the expense of allowing the constants to depend on ζ.

(ii) Bando gives a different argument for the lower bound of $\Delta\hat{u}$ making use of the fact that in his application there is a lower bound on the Ricci curvature at the initial time.

(iii) A similar smoothing argument is also used in the proof of the Moser-Trudinger inequality for Kähler-Einstein manifolds [T], [TZ1], [PSSW].

4 Proof of Theorem 1, part (i)

We provide now the proof of Theorem 1, part (i). In view of Lemma 1, it suffices to show that $\|u\|_{C^0} \to 0$ as $t \to \infty$. Recall that b is the average of u with respect to the measure $e^{-u}\omega^n$. It suffices to show that b and $\|u-b\|_{C^0}$ tend to 0 as $t \to \infty$. This is an immediate consequence of Lemmas 3 and 4 below (together with Perelman's uniform bound for $||\nabla u||_{C^0}$), so it suffices to prove these lemmas[2].

We shall need the following Poincaré-type inequality for manifolds with $c_1(X) > 0$ (see [F] or [TZ2]).

Lemma 2 *Let u satisfy the equation $R_{\bar{k}j} - g_{\bar{k}j} = -\partial_j\partial_{\bar{k}}u$. Then the following inequality*

$$\frac{1}{V}\int_X f^2 e^{-u}\omega^n \leq \frac{1}{V}\int_X |\nabla f|^2 e^{-u}\omega^n + (\frac{1}{V}\int_X f e^{-u}\omega^n)^2, \tag{4.1}$$

holds for all $f \in C^\infty(X)$.

Proof of Lemma 2: We include the easy proof for the reader's convenience. The desired inequality is equivalent to the fact that the lowest strictly positive eigenvalue μ of the following operator

$$-g^{j\bar{k}}\nabla_j\nabla_{\bar{k}}f + g^{j\bar{k}}\nabla_{\bar{k}}f\nabla_j u = \mu f, \tag{4.2}$$

with eigenfunction f satisfies $\mu \geq 1$. (Note that this operator is self-adjoint with respect to the measure $V^{-1}e^{-u}\omega^n$, and that its kernel consists of constants.) Applying $\nabla_{\bar{l}}$ and commuting $\nabla_{\bar{l}}$ through in the first term gives

$$-g^{j\bar{k}}\nabla_j\nabla_{\bar{k}}\nabla_{\bar{l}}f + R_{\bar{l}}^{\bar{p}}\nabla_{\bar{p}}f + g^{j\bar{k}}\nabla_{\bar{l}}\nabla_j u\nabla_{\bar{k}}f + g^{j\bar{k}}\nabla_{\bar{l}}\nabla_{\bar{k}}f\nabla_j u = \mu\nabla_{\bar{l}}f. \tag{4.3}$$

[2] It has been shown by H. Li [L] that $b(t_m) \to 0$ for a sequence of times $t_m \to \infty$.

Integrate now against $g^{m\bar{l}}\nabla_m f\, e^{-u}\omega^n$ and integrate by parts. In view of the fact that $R_{\bar{k}j} + \partial_j \partial_{\bar{k}} u = g_{\bar{k}j}$, we obtain

$$\int_X |\bar{\nabla}\bar{\nabla} f|^2 e^{-u}\omega^n + \int_X |\bar{\nabla} f|^2 e^{-u}\omega^n = \mu \int_X |\bar{\nabla} f|^2 e^{-u}\omega^n, \tag{4.4}$$

from which the desired inequality $\mu \geq 1$ follows at once. Q.E.D.

Henceforth we shall denote by $\|\cdot\|_{L^2}$ the L^2 norm with respect to the measure ω^n. This L^2 norm is uniformly equivalent to the L^2 norm with respect to the measure $e^{-u}\omega^n$, in view of Perelman's theorem. The following lemma holds in all generality for the Kähler-Ricci flow:

Lemma 3 *The Ricci potential $u = u(t)$ and its average $b = b(t)$ satisfy the following inequalities, where the constant C depends only on $g_{\bar{k}j}(0)$:*

(i) $0 \leq -b \leq \|u - b\|_{C^0}$;

(ii) $\|u - b\|_{C^0}^{n+1} \ \leq\ C\, \|\nabla u\|_{L^2}\, \|\nabla u\|_{C^0}^n$.

Proof of Lemma 3: First, we observe that, as a consequence of Jensen's inequality and the convexity of the exponential function,

$$b = \frac{1}{V}\int_X u\, e^{-u}\omega^n \leq\ \log\Big(\frac{1}{V}\int_X e^u e^{-u}\omega^n\Big) = 0. \tag{4.5}$$

On the other hand, e^{-u} has average 1 with respect to the measure ω^n, and thus $\sup_X u \geq 0$. Thus $-b \leq \sup_X (u - b)$, and (i) is proved.

Next, let $A = \|u - b\|_{C^0} = |u - b|(x_0)$. Then $|u - b| \geq \frac{A}{2}$ on the ball $B_r(x_0)$ of radius $r = \frac{A}{2\|\nabla u\|_{C^0}}$ centered at x_0. If $r < \rho$, where ρ is some fixed uniform radius in Perelman's non-collapsing result, then

$$\int_X (u - b)^2\omega^n \geq \int_{B_r(x_0)} \frac{A^2}{4}\omega^n\ \geq\ c\frac{A^2}{4}\left(\frac{A}{2\|\nabla u\|_{C^0}}\right)^{2n} \tag{4.6}$$

and thus

$$\|u - b\|_{C^0}^{n+1}\ \leq\ C_1 \|\nabla u\|_{C^0}^n \|u - b\|_{L^2}. \tag{4.7}$$

Applying Lemma 2, we have

$$\|u - b\|_{C^0}^{n+1} \leq\ C_1 \|\nabla u\|_{C^0}^n \|u - b\|_{L^2}\ \leq\ C_2 \|\nabla u\|_{C^0}^n \|\nabla u\|_{L^2}. \tag{4.8}$$

On the other hand, if $r > \rho$, then integrating over the ball $B_\rho(x_0)$ gives the bound $\|u - b\|_{C^0} \leq C\, \|\nabla u\|_{L^2}$, which is a stronger estimate than the one we need. Q.E.D.

Lemma 4 *Assume the Mabuchi K-energy is bounded from below on $\pi c_1(X)$. Set*

$$Y(t) = \int_X |\nabla u|^2 \omega^n = \|\nabla u\|^2_{L^2}. \tag{4.9}$$

Then for any choice of initial condition $\omega_0 \in \pi c_1(X)$, the quantity $Y(t) \to 0$ along the Kähler-Ricci flow as $t \to \infty$.

Proof of Lemma 4: The proof of this Lemma can be found in [PS], §6. We provide the short proof, for the sake of completeness. Let $\phi = \phi(t)$ be the potential of $g_{\bar{k}j}(t)$, so that $g_{\bar{k}j} = (g_0)_{\bar{k}j} + \partial_j \partial_{\bar{k}} \phi$. Then clearly $\dot{\phi} - u$ is a constant depending only on time along the Kähler-Ricci flow, and it follows immediately from the definition of the Mabuchi K-energy (1.2) that its derivative along the Kähler-Ricci flow is given by

$$\frac{d}{dt}\mathcal{M}(\phi) = -\frac{1}{V}\int_X |\nabla u|^2 \omega^n = -\frac{1}{V}Y(t). \tag{4.10}$$

Thus, since $\mathcal{M}$ is bounded from below, we have for all $T > 0$

$$\frac{1}{V}\int_0^T Y(t)dt = \mathcal{M}(\phi_0) - \mathcal{M}(\phi) \leq C, \tag{4.11}$$

and hence $Y(t)$ is integrable over $[0, \infty)$. Equivalently $\sum_{m=0}^{\infty} \int_m^{m+1} Y(t)dt < \infty$, and hence there exists $t_m \in [m, m+1)$ with $Y(t_m) \to 0$. Next, Y satisfies the following differential identity (see [PS], eq. (2.10))

$$\dot{Y} = (n+1)Y - \int_X |\nabla u|^2 R\,\omega^n - \int_X |\bar{\nabla}\nabla u|^2 \omega^n - \int_X |\nabla\nabla u|^2 \omega^n \leq C\,Y \tag{4.12}$$

for some constant C, since $|R|$ is uniformly bounded by Perelman's estimate. This implies $Y(t) \leq Y(s)\,e^{C(t-s)}$ for $t \geq s$. In particular $Y(t) \leq Y(t_m)e^{2C}$ for all $t \in [m+1, m+2)$, and hence $Y(t) \to 0$ as $t \to \infty$. The proof of Lemma 4 and hence of Theorem 1, part (i) is complete. Q.E.D.

5 Proof of Theorem 1, part (ii) and of Theorem 2

We begin by establishing Theorem 2, part (i), that is, the statement that the stability conditions (A) and (S) together imply the exponential convergence in C^∞ of $g_{\bar{k}j}(t)$ to a Kähler-Einstein metric. This is an immediate consequence of Lemmas 5 and 6 below, so it suffices to establish those two lemmas.

Lemma 5 *Assume the Mabuchi K-energy is bounded from below on $\pi c_1(X)$ and we have $\lambda_t \geq \lambda > 0$ along the Kähler-Ricci flow with initial value $(g_0)_{\bar{k}j}$. Then, the quantity*

$Y(t) = ||\nabla u||^2_{L^2}$ *tends to* 0 *exponentially, that is, there exist constants* $\mu > 0$ *and* $C > 0$ *independent of* t *so that*

$$Y(t) \ \leq \ C\, e^{-\mu t}, \qquad t \in [0,\infty). \tag{5.1}$$

Moreover,

$$\|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R-n\|_{C^0} \leq \ C\, e^{-\frac{1}{2(n+1)}\mu t}, \qquad t \in [0,\infty). \tag{5.2}$$

Proof of Lemma 5: We recall the following inequality from [PS], which holds for the Kähler-Ricci flow without any additional assumption:

$$\dot{Y} \ \leq \ -2\lambda_t Y - 2\lambda_t \mathrm{Fut}(\pi_t(\nabla^j u)) - \int_X |\nabla u|^2 (R-n)\omega^n - \int_X \nabla^j u \nabla^{\bar{k}} u (R_{\bar{k}j} - g_{\bar{k}j})\omega^n. \tag{5.3}$$

Here $\mathrm{Fut}(\pi_t(\nabla^j u))$ is the Futaki invariant, applied to the orthogonal projection $\pi_t(\nabla^j u)$ of the vector field $\nabla^j u$ on the space of holomorphic vector fields.

Now assume that the Mabuchi K-energy is bounded below. Then the Futaki invariant Fut is identically 0, and in view of Theorem 1, part (i), the preceding inequality reduces to, for t sufficiently large,

$$\dot{Y} \ \leq \ -\lambda\, Y - \int_X \nabla^j u \nabla^{\bar{k}} u (R_{\bar{k}j} - g_{\bar{k}j})\omega^n. \tag{5.4}$$

To get exponential convergence, we would like to show $| \int_X \nabla^j u \nabla^{\bar{k}} u (R_{\bar{k}j} - g_{\bar{k}j})\omega^n | \leq \frac{\lambda}{2} Y$. This would of course follow if we knew that $R_{\bar{k}j} - g_{\bar{k}j}$ were small. In the absence of such information, we shall prove something a bit weaker, but which turns out to be sufficient for our purposes. We claim that there exists $K_0 > 0$ such that

$$\dot{Y}(t) \ \leq \ -\lambda Y(t) \ + \ \frac{\lambda}{2} Y^{\frac{1}{2}}(t) \cdot \prod_{j=1}^{N} [Y(t-a_j)]^{\frac{\delta_j}{2}} \ \text{ for all } t \geq K_0, \tag{5.5}$$

where N is an integer, the a_j are non-negative integers and the δ_j are non-negative real numbers with the property that $\sum_{j=1}^N \delta_j = \ 1$.

To see this, first note that $|\nabla^j u \nabla^{\bar{k}} u (R_{\bar{k}j} - g_{\bar{k}j})| \leq |\nabla u|^2 |R_{\bar{k}j} - g_{\bar{k}j}|$, and thus

$$\left| \int_X \nabla^j u \nabla^{\bar{k}} u (R_{\bar{k}j} - g_{\bar{k}j})\omega^n \right| \ \leq \ ||\nabla u||_{C^0} (\int_X |\nabla u|^2 \omega^n)^{1/2} (\int_X |R_{\bar{k}j} - g_{\bar{k}j}|^2 \omega^n)^{1/2}. \tag{5.6}$$

However, an integration by parts shows readily that

$$\int_X |R_{\bar{k}j} - g_{\bar{k}j}|^2 \omega^n = \int_X |\partial_j \partial_{\bar{k}} u|^2 \omega^n = \int_X |\Delta u|^2 \omega^n = \int_X |R-n|^2 \omega^n, \tag{5.7}$$

and hence

$$\left| \int_X \nabla^j u \nabla^{\bar{k}} u (R_{\bar{k}j} - g_{\bar{k}j})\omega^n \right| \ \leq \ Y^{\frac{1}{2}}(t)\, \|\nabla u\|_{C^0} \|R-n\|_{L^2}. \tag{5.8}$$

We use Lemmas 1, 3 and 4 to estimate $||R-n||_{L^2}$ by $||\nabla u||_{L^2}$ and $||\nabla u||_{C^0}$ at an earlier time $t-2$. More precisely, we have, for t sufficiently large

$$\begin{aligned} \|R-n\|_{L^2}(t) &\leq \|R-n\|_{C^0}(t) \leq K\|u\|_{C^0}(t-2) \leq 2K\|u-b\|_{C^0}(t-2) \\ &\leq C\,\|\nabla u\|_{C^0}^{\frac{n}{n+1}}(t-2)\|\nabla u\|_{L^2}^{\frac{1}{n+1}}(t-2), \end{aligned} \tag{5.9}$$

so that the coefficient of $Y^{\frac{1}{2}}(t)$ on the right hand side of (5.8) can be estimated by

$$\|\nabla u\|_{C^0}\|R-n\|_{L^2} \leq C\,\|\nabla u\|_{C^0}(t)\|\nabla u\|_{C^0}^{\frac{n}{n+1}}(t-2)\|\nabla u\|_{L^2}^{\frac{1}{n+1}}(t-2). \tag{5.10}$$

We wish to iterate this estimate by applying the following bound:

$$\|\nabla u\|_{C^0}(t) \;\leq\; C_1\|u-b\|_{C^0}(t-2) \;\leq\; C_2\|\nabla u\|_{C^0}^{\frac{n}{n+1}}(t-2)\|\nabla u\|_{L^2}^{\frac{1}{n+1}}(t-2). \tag{5.11}$$

Let $A(t)=\|\nabla u\|_{L^2}(t)$ and $B(t)=\|\nabla u\|_{C^0}(t)$. Suppose $G(t)$ is a function of the form

$$G(t)=\prod_j A(t-a_j)^{\sigma_j}\prod_k B(t-b_k)^{\varepsilon_k}$$

where a_j, b_k are non-negative integers and σ_j, ε_k are non-negative real numbers. We let $\sigma=\sum\sigma_j$ and $\varepsilon=\sum\varepsilon_k$ and assume that $\sigma+\varepsilon=2$. Then (5.11) implies $G(t) \leq C\tilde{G}(t)$ where $\tilde{G}(t)=\prod_j A(t-\tilde{a}_j)^{\tilde{\sigma}_j}\prod_k B(t-\tilde{b}_k)^{\tilde{\varepsilon}_k}$ where $\tilde{\sigma}=\sum\tilde{\sigma}_j$, $\tilde{\varepsilon}=\sum_k\tilde{\varepsilon}_k$ still have the property $\tilde{\sigma}+\tilde{\varepsilon}=2$ but $\tilde{\varepsilon}=\frac{n}{n+1}\varepsilon$. Thus if we iterate, we see that $G(t)\leq C\tilde{G}(t)$ where $\tilde{\varepsilon}<1$ and $\tilde{\sigma}>1$. Setting $\delta_j=\tilde{\sigma}_j/\tilde{\sigma}$, we get

$$\|\nabla u\|_{C^0}(t)\|\nabla u\|_{C^0}^{\frac{n}{n+1}}(t-2)\|\nabla u\|_{L^2}^{\frac{1}{n+1}}(t-2) \;\leq\; H(t)\cdot\prod_j A(t-\tilde{a}_j)^{\delta_j} \tag{5.12}$$

where $H(t)=C\prod_k B(t-\tilde{b}_k)^{\tilde{\varepsilon}_k}\prod_j A(t-\tilde{a}_j)^{\tilde{\sigma}_j-\delta_j}$. Lemma 4 implies $H(t)\to 0$ as $t\to\infty$, and thus we obtain (5.5).

Let $F(t)=Re^{-\mu t}$ where $R>0$ and $\mu\in(0,1)$ are positive numbers to be chosen later. We want to show $Y\leq F$. Assume not. Since Y is bounded we may choose R sufficiently large so that for some time $t_0>K_0$ we have $Y(t)<F(t)$ for $0\leq t<t_0$ and $Y(t_0)=F(t_0)$.

We claim

$$\dot{Y}(t_0)\leq -3\mu Y(t_0). \tag{5.13}$$

If not, then $\dot{Y}(t_0)\geq -3\mu Y(t_0)=-3\mu F(t_0)$ so that

$$-3\mu F(t_0) \;\leq\; -\lambda F(t_0)+\frac{\lambda}{2}F(t_0)^{\frac{1}{2}}\prod_{j=1}^{N}[Y(t_0-a_j)]^{\frac{\delta_j}{2}} \;\leq\; -\lambda F(t_0)+\frac{\lambda}{2}\prod_{j=0}^{N}[F(t_0-a_j)]^{\frac{\delta_j}{2}}$$

where we set $a_0=0$ and $\delta_0=1$ so that $\sum_{j=0}^N\delta_j=2$. Now we have

$$(\lambda - 3\mu) R e^{-\mu t_0} \ \leq \ \frac{\lambda}{2} R e^{-\mu t_0} e^{\mu \sum a_j \delta_j / 2}.$$

This implies

$$e^{-\mu \sum a_j \delta_j / 2} \ \leq \ \frac{\lambda}{2(\lambda - 3\mu)},$$

and choosing μ sufficiently close to zero gives a contradiction. This proves (5.13).

On the other hand, from the definition of t_0, we have

$$\frac{d}{dt}\bigg|_{t=t_0} (Y - F) \geq 0,$$

and hence $\dot{Y}(t_0) \geq -\mu Y(t_0)$, which contradicts (5.13). This proves $Y \leq F$ and thus Y decays exponentially. Now Lemma 3 implies that $||u||_{C^0}$ decays exponentially which, together with Lemma 1, shows that $||R - n||_{C^0}$ decays exponentially. The proof of Lemma 5 is complete. Q.E.D.

Lemma 6 *Assume that the scalar curvature $R(t)$ along the Kähler-Ricci flow satisfies*

$$\int_0^\infty \|R(t) - n\|_{C^0} \, dt \ < \ \infty. \tag{5.14}$$

Then the metrics $g_{\bar{k}j}(t)$ converge exponentially fast in C^∞ to a Kähler-Einstein metric.

Proof of Lemma 6: The basic observation is that the integrability of $\|R - n\|_{C^0}$ over $t \in [0, \infty)$ implies a uniform bound for $\|\phi\|_{C^0}$, where ϕ is the Kähler potential. More precisely, let the potential $\phi(t)$ of $g_{\bar{k}j}(t)$ be normalized by

$$\frac{\partial}{\partial t}\phi = \log \frac{\omega^n}{\omega_0^n} + \phi + u(0), \qquad \phi|_{t=0} = c_0, \tag{5.15}$$

with the constant c_0 chosen as in [CT], [L], and [PSS], eq. (2.10). Then $g_{\bar{k}j} = (g_0)_{\bar{k}j} + \partial_j \partial_{\bar{k}} \phi$ satisfies the Kähler-Ricci flow, and Perelman's estimate for u implies $\|\dot{\phi}\|_{C^0} \leq C$.

Now we have

$$\frac{d}{dt}(\log \frac{\omega^n}{\omega_0^n}) = g^{j\bar{k}} \dot{g}_{\bar{k}j} = -(R - n). \tag{5.16}$$

Thus for any $t \in (0, \infty)$,

$$\left| \log \frac{\omega^n}{\omega_0^n} \right| = \left| \int_0^t (R - n) \, dt \right| \leq \int_0^\infty \|R - n\|_{C^0} dt < \infty. \tag{5.17}$$

On the other hand, the Kähler-Ricci flow can be rewritten as

$$\phi = -\log \frac{\omega^n}{\omega_0^n} + \dot{\phi} - u(0) \tag{5.18}$$

and thus the uniform bound for $\|\phi\|_{C^0}$ follows from the uniform bound for $|\log(\omega^n/\omega_0^n)|$ and Perelman's uniform estimate for $\|\dot{\phi}\|_{C^0}$.

The uniform boundedness of $\|\phi\|_{C^0}$ implies the uniform boundedness of $\|\phi\|_{C^k}$ for each $k \in \mathbf{N}$ (see e.g. [Y1, C, PSS, Pa]). The metrics $g_{\bar{k}j}(t)$ are all uniformly equivalent and bounded in C^∞. Thus there must exist a subsequence of times $t_m \to +\infty$ with $\phi(t_m)$ converging in C^∞ and the limit $\phi(\infty)$ is a potential for a smooth Kähler-Einstein metric. We claim that λ_t is uniformly bounded below away from zero along the flow. If not, there would be a sequence of metrics $g_{\bar{k}j}(t_l)$ along the flow with $\lambda_{t_l} \to 0$. By the estimates above, after taking a subsequence, the $g_{\bar{k}j}(t_l)$ would converge in C^∞ to a Kähler metric $g'_{\bar{k}j}$ and $0 = \lim_{l\to\infty} \lambda_{t_l} = \lambda(g'_{\bar{k}j}) > 0$ (here one can apply the argument of [PS], §4 in the special case when the complex structure is fixed) giving a contradiction. Moreover, the Mabuchi K-energy is bounded below [BM] and so we can apply Lemma 5 to see that $Y(t) = \|\nabla u\|^2_{L^2}$ decays exponentially to 0. The arguments of [PS], §3 now show that $\|\nabla u\|_{(s)}$ decay exponentially to 0 for any Sobolev norm $\|\cdot\|_{(s)}$, and hence for any norm $\|\cdot\|_{C^k}$, since the metrics $g_{\bar{k}j}(t)$ are all equivalent, and their Riemannian curvatures all uniformly bounded. But then $\|\dot{g}_{\bar{k}j}\|_{C^k} = \|R_{\bar{k}j} - g_{\bar{k}j}\|_{C^k}$ decays exponentially to 0 for any k, and hence the metrics $g_{\bar{k}j}$ converge exponentially fast to a Kähler-Einstein metric. This completes the proof of Lemma 6 and hence of part (i) of Theorem 2. Q.E.D.

Proof of Theorem 2, parts (ii) and (iii): Part (ii) follows immediately from the argument above. Part (iii) follows immediately from Part (i) and Part (ii). Q.E.D.

Proof of Theorem 1, part (ii): First observe that equation (5.11) and Lemma 1 imply

$$\|R-n\|_{C^0}(t) \ \le\ C_1\|u\|_{C^0}(t-2) \ \le 2C_1\|u-b\|_{C^0}(t-2) \ \le\ C_2\|\nabla u\|_{C^0}^{\frac{n}{n+1}}(t-2)\|\nabla u\|_{L^2}^{\frac{1}{n+1}}(t-2).$$

Equation (5.11), together with the iteration argument used in the proof of Theorem 2, show that if t is sufficiently large then

$$\|R-n\|_{C^0}(t) \ \le\ C\prod_j A(t-a_j)^{\delta_j}\prod_k B(t-b_k)^{\varepsilon_k}, \tag{5.19}$$

where $a_j, b_k, \delta_j, \varepsilon_k$ are non-negative real numbers, $\delta+\varepsilon = \sum_j \delta_j + \sum_k \varepsilon_k = 1$ and $\delta = \frac{2}{p}$. In particular, if $N = \max_j a_j$ we have

$$\|R-n\|_{C^0}(t) \ \le\ C\prod_j Y(t-a_j)^{\frac{\delta_j}{2}} \quad \text{for } t \ge N. \tag{5.20}$$

Since $\int_0^\infty Y\,dt < \infty$ if the Mabuchi K-energy is bounded below we have

$$\int_N^\infty \|R-n\|_{C^0}^p \, dt \;\leq\; C_1 \int_N^\infty \prod_j Y(t-a_j)^{\frac{p\delta_j}{2}} dt \;\leq\; C_1 \prod_j \Big(\int_N^\infty Y(t-a_j) dt \Big)^{\frac{p\delta_j}{2}} \;<\; \infty.$$

This establishes the desired inequality in Theorem 1, part (ii). Q.E.D.

6 Further results and remarks

We conclude with some further results not required for Theorems 1 and 2 but which may be of interest. We also discuss some possibilities for further developments.

(1) The average b of the Ricci potential u with respect to the volume form $e^{-u}\omega^n$ is monotone under the Kähler-Ricci flow[3],

$$\frac{d}{dt} b \geq 0. \tag{6.1}$$

To see this, we calculate using (3.1)

$$\begin{aligned}
\frac{d}{dt} b &= \frac{1}{V}\frac{d}{dt}\int_X u e^{-u}\omega^n \\
&= \frac{1}{V}\int_X (\Delta u + u - b) e^{-u}\omega^n - \frac{1}{V}\int_X u e^{-u}(\Delta u + u - b)\omega^n + \frac{1}{V}\int_X u e^{-u}\Delta u\, \omega^n \\
&= \frac{1}{V}\int_X |\nabla u|^2 e^{-u}\omega^n - \frac{1}{V}\int_X u^2 e^{-u}\omega^n + b^2 \geq 0, \qquad (6.2)
\end{aligned}$$

where for the third line we have used the equality

$$\int_X (\Delta u) e^{-u}\omega^n = \int_X |\nabla u|^2 e^{-u}\omega^n, \tag{6.3}$$

and for the last line we have used Lemma 2.

(2) It may be worth pointing out that Bando's result [B], or the L^2 result of [PS], §6, or Theorem 1 proved here, combined with Donaldson's recent result [D2],

$$\inf_{\omega\in\pi c_1(X)} \|R-n\|_{L^2}^2 \geq \sup_{\mathcal{X}} \big(-\frac{1}{N_2^2(\mathcal{X})}\mathcal{F}(\mathcal{X})\big) \tag{6.4}$$

each shows that the lower boundedness of the Mabuchi K-energy in $\pi c_1(X)$ implies the K-semistability of X. Here $\mathcal{X}$ denotes a test-configuration, and $\mathcal{F}(\mathcal{X})$ is its Futaki invariant

[3]It has recently been brought to our attention by V. Tosatti that this had also been observed earlier by Pali [Pa].

(see [D2] for the definition of $N_2(\mathcal{X}) > 0$). Indeed, by the previously mentioned results, the lower boundedness of the K-energy implies that the left hand side is 0, and hence $\mathcal{F}(\mathcal{X}) \geq 0$ for any test configuration $\mathcal{X}$, which is the definition of K-semistability [T, D1].

(3) It is natural to ask whether a converse to Bando's result (or to Theorem 1) is true, that is, whether the existence of metrics with $\|R - n\|_{C^0} \to 0$ implies the lower boundedness of the Mabuchi K-energy. This would complement well the result of Tian [T], namely that the existence of a Kähler-Einstein metric on a manifold X with $c_1(X) > 0$ and no holomorphic vector fields is equivalent to the properness of the Mabuchi K-energy (in the case of existence of holomorphic vector fields, there are some technical assumptions on the automorphism group of X, see [T], and also [PSSW]).

(4) The condition (S) in Theorem 2 can be interpreted as a stability condition in the following sense, along the lines of [PS]. Assume that there exist diffeomorphisms $F_t : X \to X$ so that $(F_t)_*(g(t))$ converges in C^∞ to a metric $\tilde{g}(\infty)$. Then if J is the complex structure of X, the pull-backs $(F_t)_*(J)$ converge also to a complex structure $J(\infty)$ (see [PS], §4). Clearly, the eigenvalues $\lambda_{\omega(t)}$ are unchanged under F_t. If they don't remain bounded away from 0 as $t \to \infty$, then the complex structure $J(\infty)$ would have a strictly higher number of independent vector fields than J. Thus the C^∞ closure of the orbit of J under the diffeomorphism group contains a complex structure different from J, and J cannot be included in a Hausdorff moduli space of complex structures.

(5) One set of assumptions which would guarantee the existence of diffeomorphisms F_t is the uniform boundedness of the Riemannian curvature tensor along the Kähler-Ricci flow. Under such an assumption, it was shown in [PS] that the condition (B) introduced there, and quoted earlier in our Introduction, implies our condition (S). Clearly, it would be very valuable to relate (B) and (S) in more general situations.

(6) Under the sole condition of lower boundedness of the Mabuchi K-energy, it was shown in [PS], §6, that there exists a sequence of times $t_m \to \infty$ with

$$\|\nabla R(t_m)\|_{L^2} \to 0. \tag{6.5}$$

It would be interesting to determine whether this convergence can take place with stronger norms.

(7) We observe that the following alternative version of Theorem 2, part (iii) also holds by our results: if a solution $g(t)$ of the Kähler-Ricci flow converges to a Kähler-Einstein metric in C^∞ *modulo automorphisms* (i.e. there exists a family of biholomorphisms $\Psi_t : X \to X$ such that $(\Psi_t)_*(g(t))$ converges in C^∞ to a Kähler-Einstein metric) then the unmodified Kähler-Ricci flow $g(t)$ converges exponentially fast to a Kähler-Einstein metric.

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